The Correct Derivation of the Cost of Equity in a MM World
(Draft. Not to be cited)

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and
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The Correct Derivation of the Cost of Equity in a MM World

Abstract
Modigliani and Miller MM (1963) say that under a perfect market situation the value of a firm remains unchanged when the capital structure changes. This is, when the proportion of debt and equity changes. When imperfections arise, such as taxes, this assertion changes. In that case, the value of the firm increases by the present value of tax savings. Tax savings are a subsidy the government gives to the taxpayers every time they spend on deductible items and pay taxes. The tax saving or tax shield is the expense times the tax rate. One of the deductible expenses is the interest charge, I = Dd, where I is the interest paid, D is the debt at the beginning of period and d is the before tax cost of debt. In that case, the tax savings are TdD.

There is a debate about how to calculate the value added by the tax savings. Some authors (MM, Myers (1974),) support the idea of d as the discounting rate for the tax savings. Others support a mixed approach: some periods the discount rate will be ρ (from 2 to n) and for period 1 the discount rate would be d see Miles and Ezzell, (1980) and (XXXX). In the body of this paper we support the idea of other authors such as Harris & Pringle (1985) and Ruback (2000) Arditti & Levi (XXXX) that say the discount rate should be ρ for every period but with restriction to debt. Tham and Vélez-Pareja (2001) have shown that the correct rate of discount for the tax savings is ρ for any amount or proportion of debt. And others (Fernandez, (1999, 2000)) support the idea that there is not a discount rate for the tax savings because the tax savings is the difference between two different cash flows (levered and unlevered) with different risk levels and it doesn’t make any sense to subtract them. Each cash flow has to be
discounted at their respective rates and after that, the DVTS is found subtracting the two PV.
All the authors say to work within a MM world. In this article we derive the value of e, the opportunity cost of equity with both assumptions: d as the discount rate and ρ as the discount rate. It is important to stress that this paper is written in a MM context.
In this paper we derive the cost of equity under different assumptions for tax savings and for the discount rate to be used for the tax savings. We show that the traditional formulation holds only for perpetuities. That with Myers (1974) APV approach the traditional formulation for period 1 and 2 do not hold. Finally it is shown that the only consistent approach is discounting tax savings (dDT) with ρ
Examples for n = 1 and n = 2 are shown.

JEL codes
D61: Cost-Benefit Analysis G31: Capital Budgeting
H43: Project evaluation

Key words or phrases
Cost of equity, discount rate for tax shield
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Introduction

Modigliani and Miller MM (1963) say that under a perfect market situation the value of a firm remains unchanged when the capital structure changes. This is, when the proportion of debt and equity changes. When imperfections arise, such as taxes, this proposition changes. In that case, the value of the firm increases by the present value of tax savings. Tax savings are a subsidy the government gives to the taxpayers every time they spend on deductible items and pay taxes. The tax saving or tax shield is the expense times the tax rate. One of the deductible expenses is the interest charge, \( I = Dd \), where \( I \) is the interest paid, \( D \) is the debt at the beginning of period and \( d \) is the before tax cost of debt. In that case, the tax savings are \( TdD \).

There is a debate about how to calculate the value added by the tax savings. Some authors (MM, Myers (1974),) support the idea of \( d \) as the discounting rate for the tax savings. Others support a mixed approach: some periods the discount rate will be \( \rho \) (from 2 to \( n \)) and for period 1 the discount rate would be \( d \) see Miles and Ezzell, (1980) and (XXXX). In this paper we partially support the idea of other authors such as Harris & Pringle (1985) and Ruback (2000) Arditti &
Levi (XXXX) that say the discount rate should be \( \rho \) for every period but with restriction to debt. Tham and Vélez-Pareja (2001) have shown that the correct rate of discount for the tax savings is \( \rho \) for any amount or proportion of debt.

All the authors work within a MM world. In this article we derive the value of \( e \), the opportunity cost of equity with both assumptions: \( d \) as the discount rate and \( \rho \) as the discount rate. We show that the traditional formulation holds only for perpetuities. We also show that Myers (1974) APV approach, the traditional formulation for period 1 and 2 does not hold.

We formulate \( e \) in a general fashion including a general definition for the rate of interest charged for debt. We show that assuming the tax savings as \( \upsilon DT \) and discounting them with \( \rho \) we obtain the correct formulation. Finally it is shown that the only consistent approach is discounting tax savings (\( \upsilon DT \)) with \( \rho \). Examples for \( n = 1 \) and \( n = 2 \) are shown.

It is worth to say that the key MM contribution hinges on the idea that \( V_L = V_{UL} + DVTS \) (Discounted Value of Tax Savings) and not in the particular result of \( DVTS = TD \), which is found when a perpetuity TdD is discounted at \( d \). Some might think that the key MM contribution is to discount the tax savings at \( d \). No and again: the crux of the MM
contribution is to show that the tax savings increases the value of the unlevered firm when the firm is levered and there exist taxes.

The summary of our findings is that the correct discount rate for the tax savings is $\rho$. The literature usually presents $d$ as the correct discount rate. However, it has to be stated that $d$ includes the risk perceived by the debt holder, not by the firm. It is not correct, then, to assume $d$ as the discount rate of the cash flow for tax savings. Earning tax savings or not, doesn't depends on the risk of the debt holder. It depends on whether the firm has enough EBIT in order to take advantage of the tax savings. The risk associated to the tax savings is the same as the risk associated to operational flows. Hence, we use $\rho$ as the correct discount rate.

In this paper we derive the formulation for $e$, the cost of equity under different assumptions regarding the discount rate for the tax savings, for perpetuity, for $n = 1$ and $n = 2$. By mathematical induction, we generalize the structure of $e$ for $n$ finite.

The only approach consistent for any case is to consider $\rho$ as the rate of discount for the tax savings.

In table 1 we show the different formulations for $e$, according to the discount rate for tax savings assumptions.
Table 1

Different formulations for $e$, the cost of equity

<table>
<thead>
<tr>
<th></th>
<th>General formulation</th>
<th>$\psi = d$ and $\upsilon = d$</th>
<th>$\psi = \rho$ and $\upsilon = d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Perpetuity</td>
<td>$e = \rho + (\rho - d)D/E^L - (\rho - \psi)VTS/E^L$</td>
<td>$e = \rho + (\rho - d)D/E^L - (\rho - d)TD/E^L$</td>
<td>$e = \rho + (\rho - d)D/E^L$</td>
</tr>
<tr>
<td>$n = 1$</td>
<td>$e = \rho + (\rho - d)D/E - (\rho - \psi)TD/(1+T)$</td>
<td>$e = \rho + (\rho - d)D/E - (\rho - d)TD/(1+T)$</td>
<td>$e = \rho + (\rho - d)D/E$</td>
</tr>
<tr>
<td>$n = 2$</td>
<td>$e = \rho + (\rho - d)D/E - (\rho - \psi)VTS/E^L$</td>
<td>$e = \rho + (\rho - d)D/E - (\rho - d)TD/(1+T)$</td>
<td>$e = \rho + (\rho - d)D/E$</td>
</tr>
<tr>
<td>$n = 2$</td>
<td>$e_1 = \rho + (\rho - d)D_0/E_{1,0}^{L}$</td>
<td>$e_1 = \rho + (\rho - d)D_0/E_{1,0}^{L}$</td>
<td>$e_1 = \rho + (\rho - d)D_0/E_{1,0}^{L}$</td>
</tr>
</tbody>
</table>

Where
- $\rho$ (rho) The (required) real return to unlevered equity and $\alpha = 1/(1 + \rho)$.
- $d$ The cost of debt (assumed constant) and $\beta = 1/(1 + d)$
- $e_n$ The return to equity (levered) in year $n$ and $\kappa_n = 1/(1 + e_n)$
- $\psi_n$ (psi) The appropriate discount rate for the tax shield in year $n$ and $\lambda_n = 1/(1 + \psi_n)$
- $\upsilon$ (upsilon) Interest rate for the calculation of the annual tax savings. Typically it is assumed to be the cost of debt, $d$. 


As can be seen in table 1, the traditional MM formulation
\[ e = \rho + (\rho - d)(1 - T)D/E \]
holds only for perpetuities. For \( n = 1 \) and \( n = 2 \) the formulation is different. Note that \( e_2 \) is the same case as \( e \) when \( n = 1 \). It is interesting to note that the structure of \( e \) is the same: the \( e \) with \( \rho \) as discount rate for the tax savings \( (e = \rho + (\rho - d)D/E) \) plus or minus something.

It can be inferred that for \( n \) finite the \( e \) will be \( e = \rho + (\rho - d)D/E \) plus something. This additional amount will vary for every \( t \). It is also interesting to observe that \( e \) is independent from the rate of interest used to calculate the tax saving, this is, \( \nu \).

In the appendix the algebra work for deriving \( e \) is presented.

**Numerical examples**

In this first example we show the total market value calculations for \( n = 1 \) and for three different \( e \) formulations:

- **Case 1.** \( e \) formulation when the discount rate for TS is \( \rho \):
  \[ e = \rho + (\rho - d)D/E \]

- **Case 2.** The traditional textbook formulation: \( e = \rho + (\rho - d)(1 - T)D/E \) derived assuming perpetuity and used for any \( n \), finite.

- **Case 3.** The formulation found in this paper for the traditional formulation and \( n = 1 \):
  \[ e = \rho + (\rho - d)(1 - T)D/E + (\rho - d)TD/((1 + d)E). \]
Let us assume we have a firm with this basic information

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Debt D</td>
<td>1,000.00</td>
</tr>
<tr>
<td>Initial equity investment, book value, E</td>
<td>500.00</td>
</tr>
<tr>
<td>Tax rate, T</td>
<td>40%</td>
</tr>
<tr>
<td>Cost of debt before taxes, d</td>
<td>12%</td>
</tr>
<tr>
<td>Unlevered cost of equity ρ</td>
<td>15%</td>
</tr>
</tbody>
</table>

The related cash flows are

<table>
<thead>
<tr>
<th>Year</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Investment</td>
<td>(1,500.00)</td>
</tr>
<tr>
<td></td>
<td>Cash flow to debt CFD</td>
<td>1,120.00</td>
</tr>
<tr>
<td></td>
<td>Cash flow to equity CFE</td>
<td>700.00</td>
</tr>
<tr>
<td></td>
<td>Tax savings TS</td>
<td>48.00</td>
</tr>
<tr>
<td></td>
<td>Free Cash flow FCF = CFD +CFE - TS</td>
<td>1,772.00</td>
</tr>
<tr>
<td></td>
<td>FCF</td>
<td>(1,500.00)</td>
</tr>
</tbody>
</table>

Case 1. Value calculation assuming ρ as the discount rate for tax savings and n = 1 and e = ρ + (ρ − d)D/E

Now we proceed to make the WACC calculations. First we calculate the contribution of debt to WACC.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>D% = D0/TV0</td>
<td></td>
</tr>
<tr>
<td>d(1-T)</td>
<td>7.20%</td>
</tr>
<tr>
<td>Contribution of D to WACC</td>
<td>4.55%</td>
</tr>
</tbody>
</table>

The contribution of equity to WACC is calculated now. The formulation for e is based on the assumption of ρ as the discount rate of the tax savings, according to Table 1.
\[ E\% = 1 - D\% \]
\[ e = \rho + (\rho - d)\frac{D}{E} \]
\[ \text{Contribution of } E \text{ to WACC} = 7.42\% \]
\[ \text{WACC} = \text{Contribution of } D \text{ plus contribution of } E = 11.97\% \]

With this WACC we calculate the total value of the firm.

| Total value | 1,582.61 |

With this total value and the value of debt, the market value of equity can be calculated as Total value minus debt.

| Equity = Total value - debt | 582.61 |

Using the CFE we can calculate the market value of equity

\[ \text{Equity} = \text{PV(CFE at } e) \]

| 582.61 |

As we assumed that the discount rate for the tax savings is \( \rho \), we calculate the discounted value of tax savings DVTS and add it up to the unlevered value of the firm.

| PV(FCF at } \rho) | 1,540.87 |
| PV(TS at } \rho) | 41.74 |
| Total | 1,582.61 |

As can be seen, total value calculated with WACC and APV and equity values coincide.

Case 2. Value calculation assuming \( d \) as the discount rate for tax savings and \( n = 1 \) and \( e = \rho + (\rho - d)(1 - T)\frac{D}{E} \).
Using the same basic information, now we proceed to make the WACC calculations. First, we calculate the contribution of debt to WACC.

| D% = D₀/TV₀ | 62.77% |
| d(1-T) | 7.20% |
| Contribution of D to WACC | 4.52% |

The contribution of equity to WACC is calculated now. The formulation for e is based on the assumption of d as the discount rate of the tax savings, according to Table 1.

| E% = 1 - D% | 37.23% |
| e = ρ + (ρ - d)(1 - T)D/E | 18.04% |
| Contribution of E to WACC | 6.71% |
| WACC = Contribution of D plus contribution of E | 11.23% |

With this WACC we calculate the total value of the firm.

| Total value | 1,593.04 |

With this total value and the value of debt, the market value of equity can be calculated as Total value minus debt.

| Equity = Total value - debt | 593.04 |

Using the CFE we can calculate the market value of equity
Equity = PV(CFE at e) \[593.04\]

As we assumed that the discount rate for the tax savings is \(d\), we calculate the discounted value of tax savings \(DVTS\) and add it up to the unlevered value of the firm.

<table>
<thead>
<tr>
<th>PV(FCF at (\rho))</th>
<th>1,540.87</th>
</tr>
</thead>
<tbody>
<tr>
<td>PV(TS at (d))</td>
<td>42.86</td>
</tr>
<tr>
<td>Total</td>
<td>1,583.73</td>
</tr>
</tbody>
</table>

Total value calculated with WACC and APV do not coincide, but equity values from CFE coincide.

Case 3. Value calculation assuming \(d\) as the discount rate for tax savings and \(n = 1\) and \(e = \rho + (\rho - d)(1 - T)D/E + (\rho - d)TD/((1 +d)E)\).

Using the same basic information, now we proceed to make the WACC calculations. First, we calculate the contribution of debt to WACC.

| D\% = D_0/TV_0 | 63.14\% |
| d(1-T)         | 7.20\%  |
| Contribution of D to WACC | 4.55\% |
The contribution of equity to WACC is calculated now. The formulation for $e$ is based on the assumption of $d$ as the discount rate of the tax savings, according to Table 1.

\[
\begin{align*}
E% &= 1 - D% \\
e &= \frac{\rho + (\rho - d)(1 - T)D/E + (\rho - d)TD/(1 + d)E}{D/E + (\rho - d)T} \\
\text{Contribution of E to WACC} &= 7.34% \\
\text{WACC} &= \text{Contribution of D plus contribution of E} \\
\end{align*}
\]

With this WACC we calculate the total value of the firm.

\[
\begin{align*}
\text{Total value} &= 1,583.73
\end{align*}
\]

With this total value and the value of debt, the market value of equity can be calculated as Total value minus debt.

\[
\begin{align*}
\text{Equity} &= \text{Total value} - \text{debt} \\
\end{align*}
\]

Using the CFE we can calculate the market value of equity

\[
\begin{align*}
\text{Equity} &= \text{PV(CFE at } e) \\
\end{align*}
\]

As we assumed that the discount rate for the tax savings is $d$, we calculate the discounted value of tax savings $DVTS$ and add it up to the unlevered value of the firm.

\[
\begin{align*}
\text{PV(FCF at } \rho) &= 1,540.87 \\
\text{PV(TS at } d) &= 42.86 \\
\text{Total} &= 1,583.73
\end{align*}
\]

Total value calculated with WACC and APV and equity values coincide.
The second example considers \( n = 2 \) and three different definitions for \( e \):

**Case 1.** \( e \) formulation when the discount rate for TS is \( \rho \):

\[
e = \rho + (\rho - d)\frac{D}{E}
\]

**Case 2.** Assuming \( d \) as the discount rate and \( e \) defined by the traditional textbook formulation \( e = \rho + (\rho - d)(1 - T)\frac{D}{E} \) derived assuming perpetuity and used for any \( n \), finite.

**Case 3.** Assuming \( d \) as the discount rate for TS and the \( e_1 \) and \( e_2 \) definitions for \( e \). We have found that

\[
e_2 = \rho + (\rho - d)(1 - T)\frac{D}{E} + (\rho - d)TD/((1 + d)E)
\]

and

\[
e_1 = \rho + (\rho - d)\frac{D_0}{E_0^F} - (\rho - d)\tau d[D_0 + D_1/(1+d)]/((1 + d) E_0^F)
\]

With \( n=2 \), we have found the following numerical results

Let us assume we have a firm with this basic information

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Tax rate, ( T )</td>
<td>35%</td>
</tr>
<tr>
<td>Cost of debt before taxes, ( d )</td>
<td>11.2%</td>
</tr>
<tr>
<td>Unlevered cost of equity ( \rho )</td>
<td>15.1%</td>
</tr>
</tbody>
</table>

The FCF is

<table>
<thead>
<tr>
<th>Year</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>FCF</td>
<td>500,000.00</td>
<td>345,000.00</td>
<td></td>
</tr>
</tbody>
</table>

The debt schedule is

\(^1\) For a finite \( n \) we need to define \( n-2 \) formulations for \( e \). From the formulation \( n-1 \) on, \( D \) and \( E \) has the meaning of absolute values. For \( n = 1 \) or perpetuities, \( D \) and \( E \) might be expressed as percentage.
The initial equity and total investment are

<table>
<thead>
<tr>
<th>Initial equity</th>
<th>125,000.00</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total initial investment</td>
<td>500,000.00</td>
</tr>
</tbody>
</table>

Case 1 Value calculation assuming \( \rho \) as the discount rate for tax savings and \( n = 2 \) and \( e = \rho + (\rho - d)D/E \).

Now we proceed to make the WACC calculations. First we calculate the contribution of debt to WACC.

<table>
<thead>
<tr>
<th>Year</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>D%</td>
<td>52.8%</td>
<td>24.8%</td>
<td></td>
</tr>
<tr>
<td>d(1-T)</td>
<td>7.3%</td>
<td>7.3%</td>
<td></td>
</tr>
<tr>
<td>Contribution of D to WACC</td>
<td>3.8%</td>
<td>1.8%</td>
<td></td>
</tr>
</tbody>
</table>

The contribution of equity to WACC is calculated now. The formulation for \( e \) is based on the assumption of \( \rho \) as the discount rate of the tax savings, according to Table 1.

<table>
<thead>
<tr>
<th>Year</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>E%</td>
<td>47.2%</td>
<td>75.2%</td>
<td></td>
</tr>
<tr>
<td>( e = \rho + (\rho - d)D/E )</td>
<td>19.5%</td>
<td>16.4%</td>
<td></td>
</tr>
<tr>
<td>Contribution of E to WACC</td>
<td>9.2%</td>
<td>12.3%</td>
<td></td>
</tr>
<tr>
<td>WACC = Contribution of D plus contribution of E</td>
<td>13.0%</td>
<td>14.1%</td>
<td></td>
</tr>
</tbody>
</table>

With this WACC we calculate the total value of the firm.

<table>
<thead>
<tr>
<th>Year</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total value</td>
<td>709,812.0</td>
<td>302,293.7</td>
<td>-</td>
</tr>
</tbody>
</table>
With this total value and the value of debt, the market value of equity can be calculated as Total value minus debt

\[
\text{Equity market value} = \text{Total value} - \text{debt} \quad 334,812.04
\]

Based on the cost of debt before taxes and the debt balance, the interest charge and tax savings are calculated.

<table>
<thead>
<tr>
<th>Year</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interest</td>
<td>42,000.0</td>
<td>8,400.0</td>
<td></td>
</tr>
<tr>
<td>Tax savings</td>
<td>14,700.0</td>
<td>2,940.0</td>
<td></td>
</tr>
</tbody>
</table>

As we assumed that the discount rate for the tax savings is \( \rho \), we calculate the discounted value of tax savings \( \text{DVTS} \) and add it up to the unlevered value of the firm.

\[
\begin{align*}
\text{PV(FCF at } \rho \text{)} & \quad 694,821.34 \\
\text{PV(TS at } \rho \text{)} & \quad 14,990.70 \\
\text{Total Value} & \quad 709,812.04 \\
\end{align*}
\]

Using the CFE we can calculate the market value of equity discounting it at \( e \).

<table>
<thead>
<tr>
<th>Year</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>CFE</td>
<td>172,700.00</td>
<td>264,540.00</td>
<td></td>
</tr>
<tr>
<td>PV(CFE at } e \text{)} &amp; \quad 334,812.0</td>
<td>227,293.7</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

As can be seen, total values and equity values coincide.

Case 2. Assuming \( d \) as the discount rate and \( e \) defined by the traditional formulation \( e = \rho + (\rho - d)(1 - T)D/E \) and \( n = 2 \)
Now we proceed to make the WACC calculations. First we calculate the contribution of debt to WACC.

<table>
<thead>
<tr>
<th>Year</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>D% = D₀/TV₀</td>
<td>52.4%</td>
<td>24.7%</td>
<td></td>
</tr>
<tr>
<td>d(1-T)</td>
<td>7.3%</td>
<td>7.3%</td>
<td></td>
</tr>
<tr>
<td>Contribution of D to WACC</td>
<td>3.8%</td>
<td>1.8%</td>
<td></td>
</tr>
</tbody>
</table>

The contribution of equity to WACC is calculated now. The formulation for e is based on the assumption of ρ as the discount rate of the tax savings, according to Table 1.

<table>
<thead>
<tr>
<th>Year</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>E% = 1 - D%</td>
<td>47.6%</td>
<td>75.3%</td>
<td></td>
</tr>
<tr>
<td>e = ρ + (ρ - d)(1 - T)D/E</td>
<td>17.9%</td>
<td>15.9%</td>
<td></td>
</tr>
<tr>
<td>Contribution of E to WACC</td>
<td>8.5%</td>
<td>12.0%</td>
<td></td>
</tr>
<tr>
<td>WACC = Contribution of D plus contribution of E</td>
<td>12.3%</td>
<td>13.8%</td>
<td></td>
</tr>
</tbody>
</table>

With this WACC we calculate the total value of the firm.

<table>
<thead>
<tr>
<th>Year</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total value</td>
<td>715,032.0</td>
<td>303,183.1</td>
<td>-</td>
</tr>
</tbody>
</table>

With this total value and the value of debt, the market value of equity can be calculated as Total value minus debt

| Equity market value = Total value - debt | 340,032.02 |
Based on the cost of debt before taxes and the debt balance, the interest charge and tax savings are calculated.

<table>
<thead>
<tr>
<th>Year</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interest</td>
<td>42,000.0</td>
<td>8,400.0</td>
<td></td>
</tr>
<tr>
<td>Tax savings</td>
<td>14,700.0</td>
<td>2,940.0</td>
<td></td>
</tr>
</tbody>
</table>

As we assumed that the discount rate for the tax savings is $d$, we calculate the discounted value of tax savings $DV_{TS}$ and add it up to the unlevered value of the firm.

\[
PV(FCF \text{ at } p) = 694,821.34 \\
PV(TS \text{ at } d) = 15,597.02 \\
\text{Total Value} = 710,418.35
\]

Using the CFE we can calculate the market value of equity discounting it at $e$.

<table>
<thead>
<tr>
<th>Year</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>CFE</td>
<td>172,700.0</td>
<td>264,540.0</td>
<td></td>
</tr>
<tr>
<td>$PV(CFE \text{ at } e) = 340,032.0</td>
<td>228,183.1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

As can be seen, total values and equity values do not coincide.

Case 3. Assuming $d$ as the discount rate for TS and the $e_1$ and $e_2$ definitions for $e$

Now we proceed to make the WACC calculations. First we calculate the contribution of debt to WACC.
The contribution of equity to WACC is calculated now. The formulation for $e$ is based on the assumption of $d$ as the discount rate of the tax savings, according to Table 1.

\[
\begin{align*}
\text{Contribution of E to WACC} & = 9.1\% \quad 12.3\%
\end{align*}
\]

With this WACC we calculate the total value of the firm.

\[
\begin{align*}
\text{Total value} & = 710,418.35 \quad 302,383.2
\end{align*}
\]

With this total value and the value of debt, the market value of equity can be calculated as Total value minus debt

\[
\text{Equity market value} = \text{Total value} - \text{debt} = 335,418.35
\]

Based on the cost of debt before taxes and the debt balance, the interest charge and tax savings are calculated.
As we assumed that the discount rate for the tax savings is \( d \), we calculate the discounted value of tax savings \( DVTS \) and add it up to the unlevered value of the firm.

\[
\begin{array}{l|c}
\text{Year} & 0 \\
\hline
\text{PV(FCF at } p) & $694,821.34 \\
\text{PV(TS at } d) & $15,597.02 \\
\text{Total Value} & $710,418.35 \\
\end{array}
\]

Using the CFE we can calculate the market value of equity discounting it at \( e \).

Using the CFE we can calculate the market value of equity discounting it at \( e \).

\[
\begin{array}{l|c|c|c}
\text{Year} & 0 & 1 & 2 \\
\hline
\text{CFE} & 172,700.00 & 264,540.00 \\
\text{PV(CFE at } e) & 335,418.4 & 227,383.2 \\
\end{array}
\]

As can be seen, now we have used the correct definition for \( e \) in the \( n=2 \) case, total values and equity values do coincide.
Consistency for n = 1

<table>
<thead>
<tr>
<th>Value calculation assuming ρ as the discount rate for tax savings, TS = TDd and e = ρ + (ρ − d)D/E</th>
<th>Total values</th>
<th>Equity values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coincide</td>
<td>Coincide</td>
<td></td>
</tr>
</tbody>
</table>

Value calculation assuming d as the discount rate for tax savings, TS = TDd and e = ρ + (ρ − d)(1 − T)D/E

<table>
<thead>
<tr>
<th>Total values</th>
<th>Equity values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Do not coincide</td>
<td>Do not coincide</td>
</tr>
</tbody>
</table>

Value calculation assuming d as the discount rate for tax savings, TS = TDd and e = ρ + (ρ − d)(1 − T)D/E + (ρ − d)TD/(1 + d)E.

<table>
<thead>
<tr>
<th>Total values</th>
<th>Equity values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coincide</td>
<td>Coincide</td>
</tr>
</tbody>
</table>

Consistency for n = 2

<table>
<thead>
<tr>
<th>Value calculation assuming ρ as the discount rate for tax savings, TS = TDd and e = ρ + (ρ − d)D/E</th>
<th>Total values</th>
<th>Equity values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coincide</td>
<td>Coincide</td>
<td></td>
</tr>
</tbody>
</table>

Value calculation assuming d as the discount rate for tax savings, TS = TDd and e = ρ + (ρ − d)(1 − T)D/E

<table>
<thead>
<tr>
<th>Total values</th>
<th>Equity values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Do not coincide</td>
<td>Do not coincide</td>
</tr>
</tbody>
</table>

Value calculation assuming d as the discount rate for tax savings, TS = TDd and e1 = ρ + (ρ - d)*D0/E0 - (ρ - d)*t*(D0 + D1)*[L0]/((1 + d)E0)

And e2 = ρ + (ρ - d)(1 - T)D/E + (ρ - d)TD/((1 + d)E).

<table>
<thead>
<tr>
<th>Total values</th>
<th>Equity values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coincide</td>
<td>Coincide</td>
</tr>
</tbody>
</table>

Both examples are available on request. The only approach that coincides in Total and Equity Value is value calculation assuming ρ as the discount rate for tax savings and e = ρ + (ρ − d)D/E. Or, calculate e for each period and with different formulation. These values of e have different formulation from the usual and traditional textbook formula.

Conclusions

Consistency among different e (cost of equity) formulations

Assumptions regarding the discount rate for tax savings TS:

<table>
<thead>
<tr>
<th>Author(s)</th>
<th>TS</th>
<th>Discount rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Modigliani &amp; Miller (M&amp;M)</td>
<td>TDd</td>
<td>d</td>
</tr>
<tr>
<td>Myers (M)</td>
<td>TDd</td>
<td>d</td>
</tr>
<tr>
<td>Harris &amp; Pringle (restricted values for D) (H&amp;P)</td>
<td>TDd</td>
<td>ρ</td>
</tr>
<tr>
<td>Ruback (restricted values for D) (R)</td>
<td>TDd</td>
<td>ρ</td>
</tr>
<tr>
<td>Tham &amp; Velez (any value for D) (TV)</td>
<td>TDd</td>
<td>ρ</td>
</tr>
</tbody>
</table>
With these assumptions \( e \) (the cost of equity) differs and the calculated WACC as well. Hence, total values differ when using WACC and APV.

### Perpetuity case

<table>
<thead>
<tr>
<th>Author</th>
<th>( e )</th>
<th>Total Value APV</th>
<th>Total Value WACC</th>
</tr>
</thead>
<tbody>
<tr>
<td>M&amp;M(d)</td>
<td>( e = \rho + (1 - T)(\rho - d)D/E )</td>
<td>Don’t Match</td>
<td>Don’t Match</td>
</tr>
<tr>
<td>M</td>
<td>( e = \rho + (1 - T)(\rho - d)D/E )</td>
<td>Don’t Match</td>
<td>Don’t Match</td>
</tr>
<tr>
<td>H&amp;P</td>
<td>( e = \rho + (\rho - d)D/E )</td>
<td>Match</td>
<td>Match</td>
</tr>
<tr>
<td>R</td>
<td>( e = \rho + (\rho - d)D/E )</td>
<td>Match</td>
<td>Match</td>
</tr>
<tr>
<td>T&amp;V</td>
<td>( e = \rho + (\rho - d)D/E )</td>
<td>Match</td>
<td>Match</td>
</tr>
</tbody>
</table>

**Formulations for perpetuities are the more common in the literature. It must be said that this is a limited or extreme case. Seldom we find perpetuities in real life situations.**

### \( n = 1 \)

<table>
<thead>
<tr>
<th>Author</th>
<th>( E )</th>
<th>Total Value APV</th>
<th>Total Value WACC</th>
</tr>
</thead>
<tbody>
<tr>
<td>M&amp;M(d)</td>
<td>( e = \rho + (\rho - d)(1 - T)D/E + (\rho - d)TD/(1 + d)E )</td>
<td>Match</td>
<td>Match</td>
</tr>
<tr>
<td>M</td>
<td>( e = \rho + (\rho - d)(1 - T)D/E + (\rho - d)TD/(1 + d)E )</td>
<td>Match</td>
<td>Match</td>
</tr>
<tr>
<td>H&amp;P</td>
<td>( E = \rho + (\rho - d)D/E )</td>
<td>Match</td>
<td>Match</td>
</tr>
<tr>
<td>R</td>
<td>( E = \rho + (\rho - d)D/E )</td>
<td>Match</td>
<td>Match</td>
</tr>
<tr>
<td>T&amp;V</td>
<td>( E = \rho + (\rho - d)D/E )</td>
<td>Match</td>
<td>Match</td>
</tr>
</tbody>
</table>

### \( n = 2 \)

<table>
<thead>
<tr>
<th>Author</th>
<th>( E )</th>
<th>Total Value APV</th>
<th>Total Value WACC</th>
</tr>
</thead>
<tbody>
<tr>
<td>M&amp;M</td>
<td>( e ) for perpetuity ( e = \rho + (1 - T)(\rho - d)D/E )</td>
<td>Don’t Match</td>
<td>Don’t Match</td>
</tr>
<tr>
<td>M&amp;M</td>
<td>( e_1 = \rho + (\rho - d)^<em>D_0/E_0^</em> - (\rho - \psi_1)^<em>\tau_1^<em>u_1^</em>[D_0 + D_1 \lambda_2]/((1 + \psi_1)E_0^</em>) ) and ( e_2 = \rho + (\rho - d)(1 - T)D/E + (\rho - d)TD/(1 + d)E )</td>
<td>Match</td>
<td>Match</td>
</tr>
<tr>
<td>M</td>
<td>Using ( e ) ( E = \rho + (1 - T)(\rho - d)D/E )</td>
<td>Don’t Match</td>
<td>Don’t Match</td>
</tr>
<tr>
<td>H&amp;P</td>
<td>( E = \rho + (\rho - d)D/E )</td>
<td>Match</td>
<td>Match</td>
</tr>
<tr>
<td>R</td>
<td>( E = \rho + (\rho - d)D/E )</td>
<td>Match</td>
<td>Match</td>
</tr>
<tr>
<td>T&amp;V</td>
<td>( E = \rho + (\rho - d)D/E )</td>
<td>Match</td>
<td>Match</td>
</tr>
</tbody>
</table>
### Any n finite

<table>
<thead>
<tr>
<th>Author</th>
<th>E</th>
<th>Total Value APV</th>
<th>Total Value WACC</th>
</tr>
</thead>
<tbody>
<tr>
<td>M&amp;M(d)</td>
<td>Not found. Using e for n=1 e = ρ + (ρ − d)(1 − T)D/E + (ρ − d)TD/((1 +d)E) or e for perpetuity e = ρ + (1 - T)(ρ - d)D/E</td>
<td>Don’t Match</td>
<td>Don’t Match</td>
</tr>
<tr>
<td>M&amp;M(ρ)</td>
<td>Not found. Using e for n=1 e = ρ + (ρ − d)(1 − T)D/E + (ρ − d)TD/((1 +d)E) or e for perpetuity e = ρ + (1 - T)(ρ - d)D/E or e = ρ + (ρ - d)D/E but, tS = TDρ</td>
<td>Don’t Match</td>
<td>Don’t Match</td>
</tr>
<tr>
<td>M</td>
<td>Not found. Using e for n=1 e = ρ + (ρ − d)(1 − T)D/E + (ρ − d)TD/((1 +d)E) or e for perpetuity e = ρ + (1 - T)(ρ - d)D/E</td>
<td>Don’t Match</td>
<td>Don’t Match</td>
</tr>
<tr>
<td>H&amp;P</td>
<td>E = ρ + (ρ − d)D/E</td>
<td>Match</td>
<td>Match</td>
</tr>
<tr>
<td>R</td>
<td>E = ρ + (ρ − d)D/E</td>
<td>Match</td>
<td>Match</td>
</tr>
<tr>
<td>T&amp;V</td>
<td>Derived (Tham &amp; Vélez-Pareja 2001) E = ρ + (ρ − d)D/E</td>
<td>Match</td>
<td>Match</td>
</tr>
</tbody>
</table>
When we use the \( e \) resulting from our calculations for \( n = 1 \), this is,
\[
e = \rho + (\rho - d)(1 - T)D/E + (\rho - d)T D/((1 + d)E)
\]
we find consistent results between \( PV(FCF \text{ at WACC}) \) and \( PV(FCF \text{ at } \rho) + PV(TS \text{ at } d) \). However, when we use the traditional textbook formulation for \( e \), there is no consistency. This simply means that the assumptions of \( \rho \) as the discount rate for the tax savings is correct, neat, clean and consistent with a MM world. More, it is the unique correct assumption for working on a MM world.

An obvious conclusion is that discounting the tax savings with \( d \), the value of the firm is overestimated.

**Bibliography**


Taggart, Jr, Robert A., 1991, *Consistent Valuation Cost of Capital*

Appendix

List of symbols
\( \rho \) The (required) real return to unlevered equity and \( \alpha = 1/(1 + \rho) \).
\( d \) The cost of debt (assumed constant) and \( \beta = 1/(1 + d) \).
\( e_n \) The return to equity (levered) in year \( n \) and \( \kappa_n = 1/(1 + e_n) \).
\( \psi_n \) The appropriate discount rate for the tax shield in year \( n \) and \( \lambda_n = 1/(1 + \psi_n) \).
\( \upsilon \) Interest rate for the calculation of the annual tax savings. Typically it is assumed to be the cost of debt, \( d \).

Derivation of \( e \) for perpetuities

\[
V^{TS} = \frac{\tau*\upsilon*D}{\psi} \quad (1a)
\]
\[
\psi*V^{TS} = \tau*\upsilon*D \quad (1b)
\]
\[
V^{UL} = \frac{FCF}{\rho} \quad (2a)
\]
\[
V^{UL} = \frac{FCF}{\rho} \quad (2b)
\]
\[
E^{L} = Z/e \quad (3a)
\]
\[
E^{L} = Z = FCF - d*D + \tau*\upsilon*D \quad (3b)
\]
\[
E^{L} = \psi*V^{TS} \quad (4a)
\]
\[
E^{L} = \psi*V^{TS} \quad (4b)
\]
\[
e*E^{L} = \rho*E^{L} + (\rho - d)*D - (\rho - \upsilon)*V^{TS} \quad (4c)
\]
\[
e = \rho + (\rho - d)*D - (\rho - \upsilon)*V^{TS} \quad (4d)
\]
Case 1
Assume $\psi = d$ and $\nu = d$

$$e = \rho + (\rho - d)\frac{D}{E_L} - (\rho - d)\frac{\tau D}{E_L} \quad (4e)$$

Case 2
Assume $\psi = \rho$ and $\nu = d$

$$e = \rho + (\rho - d)\frac{D}{E_L} \quad (4f)$$

Derivation of $e$ in the single period

The discount rate for the tax shield is $\psi$. The free cash flow (FCF) is net of tax. The cost of debt is $d$. The interest payment is equal to $\nu$ times the amount of debt $D$. The corporate tax rate is $\tau$. Later, as a special case, we can assume that $\nu = d$.

$$V^{TS} = \frac{\tau \nu D}{1 + \psi} \quad (1a)$$

$$V^{TS}(1 + \psi) = \tau \nu D \quad (1b)$$

The unlevered value is equal to the net of tax FCF discounted by $\rho$, the cost of unlevered equity.

$$V^{UL} = \frac{FCF}{1 + \rho} \quad (2a)$$

$$V^{UL}(1 + \rho) = FCF \quad (2b)$$

The levered value is equal to the sum of the unlevered value plus the present value of the tax shield.

$$V^L = V^{UL} + V^{TS} = V^{UL} + \frac{\tau \nu D}{1 + \psi} \quad (3a)$$

$$V^L = FCF + \frac{\tau \nu D \lambda}{1 + \rho} \quad (3b)$$

Multiply line 3b by $(1 + \rho)$

$$V^L(1 + \rho) = FCF + \tau \nu D \lambda (1 + \rho) \quad (4)$$

Also,

$$V^L = E^L + D \quad (5a)$$

Let the cash flow to (levered) equity (CFE) be $Z$. Then the cash flow to equity is equal to the free cash flow less the payment to debt plus the tax shield.
\[ Z = \text{FCF} - D^*(1 + d) + \tau^*\upsilon^*D^* \quad (6) \]

And \[ E^L*(1 + e) = Z \quad (7) \]

Combining line 6 and line 7, we obtain,
\[ E^L*(1 + e) + D^*(1 + d) = \text{FCF} + \tau^*\upsilon^*D^* \quad (8) \]

Substituting line 4 into line 8, we obtain that,
\[ E^L*(1 + e) + D^*(1 + d) = V_L^*(1 + \rho) + \tau^*\upsilon^*D^* \lambda^*(1 + \rho) \quad (9) \]

Simplifying line 9, we obtain
\[ e^*E^L + d^*D^* = \rho^*V_L^* + \tau^*\upsilon^*D^* - \tau^*\upsilon^*D^* \lambda^*(1 + \rho) \quad (10a) \]
\[ e^*E^L = \rho^*V_L^* - d^*D^* + \tau^*\upsilon^*(1 - \lambda + \lambda^*\rho) \quad (10b) \]
\[ e = \rho^* + (\rho - d)^*D^* + (\psi - \rho)^*\tau^*\upsilon^*D^* E \quad (10c) \]

We know that:
\[ (1 - \lambda) = \psi/(1 + \psi) \quad (11a) \]
\[ (1 - \lambda + \lambda^*\rho) = \psi - \rho \quad (11b) \]

Substituting line 11b into line 10c, we obtain
\[ e = \rho^* + (\rho - d)^*D^* + (\psi - \rho)^*\tau^*\upsilon^*D^* \quad (12a) \]
\[ e = \rho^* + (\rho - d)^*D^* - (\rho - \psi)^*V^{TS} \quad (12b) \]

\[ e = \rho^* + (\rho - d)^*D^* \quad (13) \]

Case 1: Assume \( \psi = \rho \).
Case 2: Assume $\psi = d$.

$$e = \rho + (\rho - d)^*D - (\rho - d)^*\tau^*\upsilon^*D$$

$$\frac{E}{1 + d}$$

(14)

Case 3: Assume that $\upsilon = d$ and $\psi = d$.

$$e = \rho + (\rho - d)^*D - (\rho - d)^*\tau^*\upsilon^*D$$

$$\frac{E}{1 + d}$$

(15a)

$$e = \rho + (\rho - d)^*D - (\rho - d)^*\tau^*\upsilon^*D$$

$$\frac{E}{1 + d}$$

(15b)

$$e = \rho + (\rho - d)^*D\{1 - \tau^*\upsilon^*\beta\}$$

$$\frac{E}{1 + d}$$

(15c)

$$e = \rho + (\rho - d)^*D\{(1 - \tau) + (1 - d^*\beta)^*\tau\}$$

$$\frac{E}{1 + d}$$

(15d)

$$e = \rho + (\rho - d)^*(1 - \tau)^*D + (\rho - d)^*\tau^*D$$

$$\frac{E}{1 + d}$$

(15d)

Derivation of $e$ for $N = 2$

$$V^L_1 = V^{UL}_1 + V^{TS}_1 = E^L_1 + D_1$$

(1a)

$$V^L_0 = V^{UL}_0 + V^{TS}_0 = E^L_0 + D_0$$

(1b)

$$V^{TS}_1 = \tau^*\upsilon^*D_1^*\lambda_2$$

(2a)

$$V^{TS}_1*(1 + \psi_2) = \tau^*\upsilon^*D_1$$

(2b)

$$V^{TS}_0 = \tau^*\upsilon^*D_0^*\lambda_1 + \tau^*\upsilon^*D_1^*\lambda_1^*\lambda_2$$

(2b)

$$V^{TS}_0 = \tau^*\upsilon^*\lambda_1^*[D_0 + D_1^*\lambda_2]$$

(2b)

$$V^{TS}_0 = \lambda_1^*\{\tau^*\upsilon^*D_0 + V^{TS}_1\}$$

(2b)

$$V^{TS}_0*(1 + \psi_1) = \tau^*\upsilon^*D_0 + V^{TS}_1$$

(2b)

$$V^{UL}_1 = FCF_2^*\alpha$$

(3a)

$$V^{UL}_0 = FCF_1^*\alpha + FCF_2^*\alpha^2$$

(3b)

$$V^{UL}_0 = \alpha^*[FCF_1 + FCF_2^*\alpha]$$

(3c)

$$V^{UL}_0*(1 + \rho) = FCF_1 + V^{UL}_1$$

(3d)

$$E^L_1 = Z_2^*\kappa_2$$

(4a)

$$E^L_1 = \{FCF_2 - D_1^*(1 + d) + \tau^*\upsilon^*D_1\}^*\kappa_2$$

(4b)

$$E^L_1*(1 + e_2) = FCF_2 - D_1^*(1 + d) + \tau^*\upsilon^*D_1$$

(4b)
$$E^L_1(1 + e_2) = V^{UL}_1(1 + \rho) - D_1(1 + d) + V^{TS}_1 + \psi_2 V^{TS}_1$$  (5a) \\
$$e_2 E^L_1 = \rho V^{UL}_1 - d D_1 + \psi_2 V^{TS}_1$$  (5b) \\
$$e_2 E^L_1 = \rho [E^L_1 + D_1 - V^{TS}_1] - d D_1 + \psi_2 V^{TS}_1$$  (5c) \\
$$e_2 E^L_1 = \rho E^L_1 + (\rho - d) D_1 + \psi_2 V^{TS}_1 - \rho V^{TS}_1$$  (5d) \\
$$e_2 = \rho + (\rho - d) D_1 + (\psi_2 - \rho) V^{TS}_1$$  (5e) \\
$$E^L_0 = Z_1 \kappa_1 + Z_2 \kappa_1 \kappa_2$$  (6a) \\
$$E^L_0 = \kappa_1 \{Z_1 + Z_2 \kappa_2\}$$  (6b) \\
$$E^L_0(1 + e_1) = [Z_1 + E^L_1]$$  (6c) \\
$$E^L_0(1 + e_1) = [FCF_1 - D_0(1 + d) + \tau \upsilon D_0] + E^L_1$$  (7a) \\
$$E^L_0(1 + e_1) = [V^{UL}_0(1 + \rho) - V^{UL}_1] - D_0(1 + d) + D_1 + [V^{TS}_0(1 + \psi_1) - V^{TS}_1] + E^L_1$$  (7b) \\
$$E^L_0(1 + e_1) = V^{UL}_0(1 + \rho) - D_0(1 + d) + V^{TS}_0(1 + \psi_1) - V^{UL}_1 + D_1 - V^{TS}_1 + E^L_1$$  (7c) \\
$$e_1 E^L_0 + E^L_0 = \rho V^{UL}_0 - d D_0 + \psi_1 V^{TS}_0 + V^{UL}_0 - D_0 + V^{TS}_0 - V^{UL}_1 + D_1 - V^{TS}_1 + E^L_1$$  (7d) \\
$$e_1 E^L_0 = \rho V^{UL}_0 - d D_0 + \psi_1 V^{TS}_0$$  (7e) \\
$$e_1 E^L_0 = \rho [V^{UL}_0 - V^{TS}_0] - d D_0 + \psi_1 V^{TS}_0$$  (7f) \\
$$e_1 E^L_0 = \rho V^{UL}_0 - d D_0 - (\rho - \psi_1) V^{TS}_0$$  (7g) \\
$$e_1 = \rho + (\rho - d) D_0 - (\rho - \psi_1) V^{TS}_0$$  (7h) \\
$$e_1 = \rho + (\rho - d) D_0 - (\rho - \psi_1) V^{UL}_0 + \psi_1 V^{TS}_0 [D_0 + D_1 \lambda_2]$$  (7i) \\
$$e_1 = \rho + (\rho - d) D_0 - (\rho - \psi_1) V^{UL}_0 [D_0 + D_1 \lambda_2]$$  (7j) \\

Case 1

For finite N

$$V^{TS}_0 = \tau \upsilon D_0 \lambda_1 + \tau \upsilon D_1 \lambda_1 \lambda_2 + \ldots$$
\[ + \tau^* \upsilon^* D_{N-2} \lambda_1 \cdots \lambda_{N-1} + \tau^* \upsilon^* D_{N-1} \lambda_1 \cdots \lambda_N \] (2)

Assume that \( \psi \) is constant. That is,
\[ \psi_1 = \psi_2 = \cdots = \psi_{N-1} = \psi_N = \psi \] (3a)

Then,
\[ \lambda_1 = \lambda_2 = \cdots = \lambda_{N-1} = \lambda_N = \lambda \] (3b)

Also, assume that the amount of the debt is constant. That is,
\[ D_0 = D_2 = \cdots = D_{N-1} = D_N = D \] (4)

Then, substituting line 3b and line 4 into line 2, we obtain that,
\[ V_{TS}^0 = \tau^* \upsilon^* D \left( \lambda + \lambda^2 + \cdots + \lambda^{N-1} + \lambda^N \right) \] (5)

\[ V_{TS}^0 = \tau^* \upsilon^* D \lambda (1 - \lambda^{N+1}) \] (6)

\[ 1 - \lambda = 1 - 1/(1 + \psi) = \psi/(1 + \psi) \] (7a)

\[ \lambda/(1 - \lambda) = \frac{1 + \psi}{1 + \psi} \frac{1}{\psi} = 1 \] (7b)

\[ V_{TS}^0 = \frac{\tau^* \upsilon^* D \psi}{\psi} (1 - \lambda^{N+1}) \] (6)

\[ e = \rho + (\rho - d) D - (\rho - \psi) \frac{V_{TS}^0}{E^L} \] (7)

\[ e = \rho + (\rho - d) D - (\rho - \psi) \frac{\tau^* \upsilon^* D (1 - \lambda^{N+1})}{E^L} \] (8)

**Case 1:** Assume that \( \upsilon = d \) and \( \psi = d \). Then
\[ e = \rho + (\rho - d) D - (\rho - d) \frac{\tau^* d^* D (1 - \lambda^{N+1})}{d \ E^L} \] (9)

\[ e = \rho + (\rho - d) D^* \{1 - \tau^*(1 - \lambda^{N+1})\} \] (10)

And
\[ \lambda^{N+1} = [(1/(1 + d))]^{N+1} \rightarrow 0 \text{ for large } N \] (11)
\[ e = \rho + (\rho - d) \frac{D}{E_L} \]  

(12)

**Summary**

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<th>( \psi = d ) and ( \psi = d )</th>
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